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Nonlinear wave propagation on an arbitrary steady transonic flow

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Here, we have studied the propagation of an arbitrary disturbance bounded in space on an arbitrary two- or three-dimensional transonic flow. First we have presented a general theory valid for an arbitrary system of n first-order quasi-linear partial differential equations and then used the theory for the special case of gasdynamic equations. If a disturbance is created in the neighbourhood of a sonic point, only a part of the disturbance stays in the transonic region and it is bounded by a wave front perpendicular to the streamlines. This part of the disturbance is governed by a very simple partial differential equation and the problem essentially reduces to the discussion of one-dimensional waves. The disturbance decays in the neighbourhood of the points where the flow accelerates from a subsonic state to a supersonic state and it attains a steady state where the flow is decelerating.

1. Introduction

Both the exact and approximate solutions of two-dimensional steady equations of motion of a polytropic gas have established beyond doubt the theoretical existence of continuously accelerating and decelerating, isentropic, mixed transonic flows past solid boundaries or through the converging or diverging nozzles. Theoretically then, a fluid element should be able to accelerate continuously from a subsonic state to a supersonic state and vice versa. Continuous flows accelerating steadily through the speed of sound could always be obtained but most of the early experiments showed that a shock wave necessarily appears where a continuously decelerating flow should exist according to theoretical predictions. Kantrowitz (1947) studied the stability of quasi-one-dimensional steady transonic flows by superposing unsteady disturbances and came to the definite conclusion that a flow in a Laval nozzle continuously accelerating through the speed of sound is stable but the reversed flow is unstable and the downstream part of latter will be replaced by an accelerating flow terminated by a shock wave. A study of the stability of two-dimensional plane flows past an aerofoil surface was done by Kuo (1951) and his conclusion that two- dimensional flows decelerating continuously through the speed of sound are also unstable was not in contradiction with the early experimental results of his time. The

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faith in the instability of a continuously decelerating transonic flow grew stronger when other arguments (Spee 1971) such as the non-existence of neighbouring flows (Morawetz 1964) poured in. However, Pearcy (1962) evolved the concept of a peaky pressure distribution and showed that for all practical purposes shock-free flow around aerofoils could be experimentally realized. Recent theoretical and experimental investigations at NLR (Nieuwland 1966; Nieuwland & Spee 1968; Spee 1971) have shown that two-dimensional transonic continuously decelerating flows are not unstable in the strict sense and it will be possible to obtain these flows experimentally as closely as possible if we can reduce model imperfections and boundary-layer effects. Spee has stressed that Kuo's approximate analysis is very similar to the one-dimensional case of Kantrowitz, and if essential two-dimensional effects are considered, the upstream-moving waves originating in the subsonic region near the tail (or in the wake) are no longer trapped in the supersonic region but penetrate through the decelerating part of the flow owing to the turning effect and thus move out of the transonic region. The reason for the turning of the wave front is the variation in the flow variables normal to the surface of the aerofoil. From their work on the stability of almost shock-free decelerating flows, Nieuwland & Spee (1968) conclude that the old 'transonic controversy' can be regarded as definitely settled. However, there are possibilities of trapped pulses in the transonic region, especially if the origin of the disturbance is near the sonic line (Spee 1971, figure 18). Even if the pulse is not trapped the detailed history of the pulse and nonlinear steepening have not been fully discussed. For this quantitative discussion, we need a 'local' approximation of the differential equations in the neighbourhood of a sonic point. The numerical solution of Spee for the location of the wave front does give a good qualitative picture of the unsteady waves on a given transonic flow but the present state of knowledge in the subject seems to be incomplete.

In the present paper we have tried to give a unified theory for two- and three-dimensional flows by deriving an approximate equation which governs the propagation of pulses moving slowly on a transonic flow. We first find that the results of Kantrowitz can be easily deduced from the general theory of Kulikovskii & Slobodkina (1967) on the equilibrium of one-dimensional steady solutions at singular points, the unsteady solutions being governed by a general system of quasi-linear partial differential equations in two independent variables. We first extend Kulikovskii & Slobodkina's theory for one-dimensional steady solutions to multi-dimensional steady solutions of a general system of quasi-linear equations and then use it to study wave propagation on two- or three-dimensional steady transonic flows. If a disturbance is created in the transonic region, only a part of the disturbance bounded by a wave front normal to the streamlines stays in the transonic region. This part of the disturbance is governed by a very simple partial differential equation and the problem essentially reduces to the discussion of one-dimensional waves and does not differ significantly from that of Kantrowitz. The approximate equation depends on the steady flow through a single parameter proportional to the acceleration of the fluid elements when they cross the sonic speed. The disturbance decays when flow is accelerating and it attains a steady state when it is decelerating. This shows that even in the present theory we have not been able to take two- and three-dimensional effects properly into account, but a systematic derivation of the approximate equation shows that quasi-one-dimensional waves are important, at least in the neighbourhood of the points on the sonic surface and for local disturbances bounded in space. The turning effect discussed by Spee is not a local property. We can have only 'almost' shock-free transonic flows in experiments and it is impossible to get rid of the weak shocks. This is explained by the present theory of local waves. A wave which originates in the wake in the subsonic flow downstream is certainly not local. Another point which we wish to emphasize is that, when the disturbances are created unintentionally in the wake or near the sonic line, they will be random in nature, i.e. the distribution of ω (see § 3) with ξ will be such that the algebraic sum of the areas of all disturbances in the ξ, ω plane will be almost zero. Since our analysis shows that equal positive and negative areas of disturbances cancel each other and the original steady flow is recovered, we conclude that a continuous decelerating flow is stable, but very weak shocks may be present at random. We also believe that we shall be able to take two- and three-dimensional effects fully into account in our future work.

In almost all theoretical investigations on transonic flows, the assumptions that the motion is isentropic and irrotational have been made. Here, we proceed with original equations without making any of these assumptions anywhere in our analysis. Throughout this paper, we use the convention that a repeated suffix in any term represents the sum over the spectrum of the suffix. We assume the spectrum of the suffixes i, j and k to be $1, 2, \ldots, n$; of α and β to be 1, 2, 3 and of p to be 1 and 2.

2. Equations of motion and the pulse geometry

If q_1, q_2 and q_3 are the three components of the particle velocity, p the pressure, ρ the mass density, x_1, x_2 and x_3 the three spatial co-ordinates and t the time, the equations of motion of an inviscid non-conducting polytropic gas are

$$A_{ij}\frac{\partial u_j}{\partial t}+B_{ij}^{(\alpha)}\frac{\partial u_j}{\partial x_\alpha} = 0 \quad (i, j = 1,2,3,4,5; \alpha = 1,2,3), \tag{2.1}$$

where
$$A_{ij} = \delta_{ij}, \quad u_1 = q_1, u_2 = q_2, \quad u_3 = q_3, \quad u_4 = p, \quad u_5 = \rho, \tag{2.2}$$

$$[B_{ij}^{(\alpha)}] = \begin{bmatrix} q_\alpha & 0 & 0 & \rho^{-1}\delta_{1\alpha} & 0 \\ 0 & q_\alpha & 0 & \rho^{-1}\delta_{2\alpha} & 0 \\ 0 & 0 & q_\alpha & \rho^{-1}\delta_{3\alpha} & 0 \\ \rho a^2\delta_{1\alpha} & \rho a^2\delta_{2\alpha} & \rho a^2\delta_{3\alpha} & q_\alpha & 0 \\ \rho\delta_{1\alpha} & \rho\delta_{2\alpha} & \rho\delta_{3\alpha} & 0 & q_\alpha \end{bmatrix} \tag{2.3}$$

and δ_{ij} is the Kronecker delta. For a polytropic gas, the local speed of sound a is given by

$$a^2 = \gamma p/\rho, \tag{2.4}$$

where the constant γ is the ratio of the specific heats.

If we create a continuous disturbance bounded by a wave front whose normal is given by the unit vector $\mathbf{n}$ (n_1, n_2, n_3), the velocity of the front (i.e. the characteristic velocity) is a root of the characteristic equation in λ:

$$|n_\alpha B_{ij}^{(\alpha)} - \lambda A_{ij}| = 0. \tag{2.5}$$

We select the unit normal in such a manner that the angle between the vectors $\mathbf{n}$ and $\mathbf{q}$ is less than or equal to $\frac{1}{2}\pi$. Since there are five roots of (2.5), any disturbance will break up into five modes and the part of the disturbance moving upstream in the flow will move with the velocity $\lambda = c$, where

$$c = n_1 q_1 + n_2 q_2 + n_3 q_3 - a. \tag{2.6}$$

The corresponding bicharacteristic velocity components, denoted by χ_α, are given by

$$dx_\alpha/dt \equiv \chi_\alpha = q_\alpha - a n_\alpha. \tag{2.7}$$

Let us consider a steady flow given by

$$u_i = u_{i0}(x_\alpha) \tag{2.8}$$

and let us assume that the state at the point $\mathbf{x}^*$ is sonic. Then $\mathbf{x}^*$ can be an arbitrary point on the sonic surface and the value c^* of c at $\mathbf{x}^*$ is

$$c^* = n_1 q_1^* + n_2 q_2^* + n_3 q_3^* - a^*, \tag{2.9}$$

where

$$a^{*2} = q_1^{*2} + q_2^{*2} + q_3^{*2}. \tag{2.10}$$

In $\mathbf{n}$ space, the equation $c^* = 0$ represents a plane passing through the point $\mathbf{n} = a^{*-1}\,\mathbf{q}^*$ and $1 - n_\alpha(q_\alpha^*/a)$ represents the length of the perpendicular drawn from $\mathbf{n}$ to the plane. Since $\mathbf{n}$ lies on the unit surface about the origin, it follows that c^* is negative for all values of $\mathbf{n}$ except for $\mathbf{n} = a^{*-1}\mathbf{q}^*$, where it attains its maximum value, zero. As we assume the angle between the two vectors $\mathbf{n}$ and $\mathbf{q}$ to be less than or equal to $\frac{1}{2}\pi$, we can easily show that no other root of the characteristic equation (2.5) vanishes at a sonic point. We therefore conclude that the part of the disturbance which can stay in a small neighbourhood of a sonic surface for a time interval of the order of unity must be bounded by a wave front normal to the streamlines, since it is only for a wave front with normal $\mathbf{n} = a^{*-1}\mathbf{q}^*$ that the front velocity c vanishes at a sonic point. As the states at different points of a disturbance propagate with bicharacteristic velocity χ and since $\chi_\alpha^* = 0$ for $n_\alpha = q_\alpha^*/a^*$, it follows that this part of the disturbance actually stays in the small neighbourhood of a sonic point for a time interval of the order of unity and must be responsible for any instability of a continuous flow which is not obtained experimentally. The vanishing of all components of the bicharacteristic velocity is, perhaps, the most important property of a point on a sonic surface.

In order to see the effect of the geometry of the front on these trapped waves, we consider a two-dimensional uniform sonic flow in the positive-x_1 direction and a disturbance which may be either on the left or on the right of a slightly curved wave front. A point on the wave front where the normal has a greater inclination to the x_1 axis will move with a greater speed in the negative-x_1 direction. Therefore, the wavefront convex to the sonic flow will ultimately become plane and

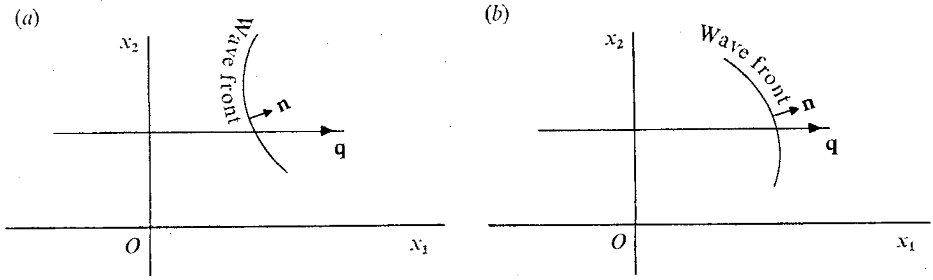

FIGURE 1. Disturbances on uniform sonic flow in x_1 direction. (*a*) A wave front convex to the sonic flow tends to become plane and perpendicular to the streamlines. (*b*) A wave front concave to the flow tends to become more curved.

perpendicular to the streamlines (figure 1*a*). On the other hand, the wave front concave to the flow (figure 1*b*) will become more curved and will move away from its place of origin. In the second case it is difficult to predict the full history and, probably, it is not important for us as the wave front will not remain for a sufficiently long time in a small transonic region.

We also note that the change in the velocity c^* due to small changes in n_α from the values q_α^*/a^* is of second order in $n_\alpha - q_\alpha^*/a^*$. Therefore, in the following discussion of the stability of transonic flows, if we neglect second-order terms we can assume that the wave front is plane and perpendicular to the streamlines.

3. Derivation of the approximate equation in the neighbourhood of a critical point

It is possible to derive the approximate equation governing the propagation of 'transonic pulses' for a more general system of equations than (2.1) without introducing any extra complication and, therefore, we shall consider here a general system of n first-order quasi-linear partial differential equations in four independent variables t, x_1, x_2, and x_3:

$$A_{ij}\frac{\partial u_j}{\partial t} + B_{ij}^{(\alpha)}\frac{\partial u_j}{\partial x_\alpha} + C_i = 0 \quad (i, j = 1, 2, \ldots, n; \alpha = 1, 2, 3), \tag{3.1}$$

where A_{ij}, $B_{ij}^{(\alpha)}$ and C_i are functions of x_α and u_i and do not depend on the time t explicitly. Let us consider a known steady solution

$$u_i = u_{i0}(x_\alpha) \tag{3.2}$$

of equations (3.1) and let $\mathbf{x}^*$ be a fixed point of the space. We denote the value of a quantity Q in the steady solution (3.2) at the point $\mathbf{x}^*$ by Q^*. Thus

$$u_i^* = u_{i0}(x_\alpha^*), \quad A_{ij}^* = A_{ij}\{u_{k0}(x_\alpha^*), x_\beta^*\}, \text{etc.} \tag{3.3}$$

We consider a disturbance of sufficiently small amplitude and bounded by a plane wave front in the neighbourhood of the point $\mathbf{x}^*$, the normal of the wave front being given by the unit vector $\mathbf{n}$. The velocity of the wave front is a solution of the characteristic equation (2.5), which gives, in general, n values of λ. We now make two assumptions about the system (3.1).

Assumption 1. The characteristic equation (2.5) has a root $\lambda = c(u_i, x_\alpha)$ which is simple and real.

This implies that the rank of the matrix

$$[n_\alpha B_{ij}^{(\alpha)} - cA_{ij}] \tag{3.4}$$

is $n-1$ and there exist unique (except for a constant factor) left and right eigenvectors $\mathbf{l} \equiv [l_1, l_2, \ldots, l_n]$ and $\mathbf{r} \equiv [r_1, r_2, \ldots, r_m]$ satisfying

$$l_i n_\alpha B_{ij}^{(\alpha)} = cl_i A_{ij}, \quad n_\alpha B_{ij}^{(\alpha)} r_j = cA_{ij} r_j. \tag{3.5}$$

No assumption has been made about any other root of (2.5). The assumption that c is a simple root has been made in order to avoid the complications of a more general theory. We can easily extend the analysis to the case where its multiplicity s is greater than one (Bhatnagar & Prasad 1971).

By the lemma on bicharacteristic directions (Courant & Hilbert 1962, p. 597), the component χ_α of the bicharacteristic velocity in the x_α direction is given by

$$\chi_\alpha = (l_i B_{ij}^{(\alpha)} r_j)/(l_i A_{ij} r_j). \tag{3.6}$$

Assumption 2. Each component of the bicharacteristic velocity vanishes in the steady solution at the point $\mathbf{x}^*$, i.e.

$$\chi_\alpha^* = 0 \quad \text{or} \quad l_i^* B_{ij}^{(\alpha)*} r_j^* = 0.$$

We define the 'critical point' in a steady solution as a point where assumption 2 is satisfied.

Since the states at different points of a disturbance in a wave motion are actually carried along the bicharacteristics, we expect the value c^* of the front velocity to be zero. We can easily verify this from the second assumption and the relation

$$c = (l_i n_\alpha B_{ij}^{(\alpha)} r_j)/(l_i A_{ij} r_j). \tag{3.7}$$

We introduce a new set of independent variables $(t', \xi, \eta_1, \eta_2)$ by defining

$$t' = t, \quad \xi = n_\alpha(x_\alpha - x_\alpha^*), \quad \eta_p = a_\alpha^{(p)}(x_\alpha - x_\alpha^*), \quad p = 1, 2, \tag{3.8}$$

where the matrix

$$\mathbf{N} = \begin{bmatrix} n_1 & n_2 & n_3 \\ a_1^{(1)} & a_2^{(1)} & a_3^{(1)} \\ a_1^{(2)} & a_2^{(2)} & a_3^{(2)} \end{bmatrix}$$

is orthogonal. At any instant the wave front is now represented by ξ = constant. The axes η_1 and η_2 lie in a plane parallel to the wave front. The system (3.1) reduces to

$$A_{ij}\frac{\partial u_j}{\partial t'} + B_{ij}\frac{\partial u_j}{\partial \xi} + D_{ij}^{(p)}\frac{\partial u_j}{\partial \eta_p} + C_i = 0, \tag{3.9}$$

where

$$B_{ij} = n_\alpha B_{ij}^{(\alpha)}, \quad D_{ij}^{(p)} = a_\alpha^{(p)} B_{ij}^{(\alpha)}. \tag{3.10}$$

The behaviour of the perturbations

$$v_k = u_k(x_\alpha, t) - u_{k0}(x_\alpha) \tag{3.11}$$

of the steady solution is described by the system

$$A_{ij}(u_k, x_\alpha)\frac{\partial v_j}{\partial t'} + B_{ij}(u_k, x_\alpha)\frac{\partial v_j}{\partial \xi} + D_{ij}^{(p)}(u_k, x_\alpha)\frac{\partial v_j}{\partial \eta_p} + F_i = 0, \tag{3.12}$$

where

$$F_i = B_{ij}(u_k, x_\alpha)\frac{\partial u_{j0}}{\partial \xi} + D_{ij}^{(p)}(u_k, x_\alpha)\frac{\partial u_{j0}}{\partial \eta_p} + C_i(u_k, x_\alpha). \tag{3.13}$$

We assume each of the perturbations v_k to be a small quantity of order δ. Expanding the various terms on the right-hand side of (3.13) and using the equations

$$B_{ij}(u_{k0}, x_\alpha)\frac{\partial u_{j0}}{\partial \xi} + D_{ij}^{(p)}(u_{k0}, x_\alpha)\frac{\partial u_{j0}}{\partial \eta_p} + C_i(u_{k0}, x_\alpha) = 0 \tag{3.14}$$

we get in a neighbourhood (with linear dimensions of order δ) of the point $\mathbf{x}^*$

$$F_i = F^*_{iv_k} v_k + O(\delta^2), \tag{3.15}$$

where

$$F^*_{iv_k} = \left(\frac{\partial B_{ij}}{\partial u_k}\right)^* \left(\frac{\partial u_j}{\partial \xi}\right)^* + \left(\frac{\partial D_{ij}^{(p)}}{\partial u_k}\right)^* \left(\frac{\partial u_j}{\partial \eta_p}\right)^* + \left(\frac{\partial C_i}{\partial u_k}\right)^*. \tag{3.16}$$

Since we wish to study waves for which the wave fronts remain in the neighbourhood of the plane $\xi = 0$ for a time interval of order unity, we need to approximate equations (3.12) over a domain in which $\xi = O(\delta)$ and each of t' and η_p is of the order of unity. Using

$$\xi' = \delta^{-1}\xi \tag{3.17}$$

we write (3.12) in the form

$$A_{ij}\frac{\partial v_j}{\partial t'} + \frac{1}{\delta}B_{ij}\frac{\partial v_j}{\partial \xi'} + D_{ij}^{(p)}\frac{\partial v_j}{\partial \eta_p} + F_i = 0. \tag{3.18}$$

Substituting

$$v_k = v_k^{(0)} + \delta v_k^{(1)} + \ldots \tag{3.19}$$

in (3.18), expanding various terms and equating terms of orders unity and δ we get

$$B^*_{ij}\,\partial v_j^{(0)}/\partial \xi' = 0 \tag{3.20}$$

and

$$B^*_{ij}\frac{\partial v_j^{(1)}}{\partial \xi'} + A^*_{ij}\frac{\partial v_j^{(0)}}{\partial t'} + \frac{1}{\delta}\Delta B_{ij}\frac{\partial v_j^{(0)}}{\partial \xi'} + D_{ij}^{(p)*}\frac{\partial v_j^{(0)}}{\partial \eta_p} + F^*_{iv_k} v_k = 0, \tag{3.21}$$

where ΔB_{ij} represents the first term in the expansion of $B_{ij}(u_{k0} + v_k, x_\alpha) - B^*_{ij}$.

The velocity c vanishes in the steady solution at $\xi = 0$, $\eta_p = 0$; and we can expand it in the form

$$c = c^{(1)} + c^{(2)}\delta + \ldots, \tag{3.22}$$

where $c^{(1)}$, $c^{(2)}$ etc. are of order δ. Expanding both sides of (3.7) we get

$$c^{(1)} = \{l^*_i(\Delta B_{ij})\, r^*_j\}/(l^*_i A^*_{ij} r^*_j). \tag{3.23}$$

The general solution of equations (3.20) is

$$v_j^{(0)} = \omega(t', \xi, \eta_p)\, r^*_j + g_j(t', \eta_p), \tag{3.24}$$

where ω and g_i are arbitrary functions of their arguments. Substituting (3.24) in (3.21), multiplying the resultant by l_i^*, using assumption 2 and dividing by $l_i^* A_{ij}^* r_j^*$ we get

$$\frac{\partial \omega}{\partial t'} + c^{(1)} \frac{\partial \omega}{\partial \xi} = K\omega + f(t', \eta_p), \tag{3.25}$$

where
$$K = -(l_i^* F_{iv_j}^* r_j^*)/(l_i^* A_{ij}^* r_j^*) \tag{3.26}$$

and
$$f(t', \eta_p) = -\frac{1}{l_i^* A_{ij}^* r_j^*} \left\{ l_i^* A_{ij}^* \frac{\partial g_j}{\partial t'} + l_i^* D_{ij}^{(p)*} \frac{\partial g_j}{\partial \eta_p} + l_i^* F_{iv_j}^* g_j \right\}. \tag{3.27}$$

From (3.23) we get

$$c^{(1)} = c_\xi \xi + c_{\eta_p} \eta_p + c_\omega \omega + \phi(t', \eta_p), \tag{3.28}$$

where
$$c_\xi = \frac{1}{l_i^* A_{ij}^* r_j^*} \left[l_i^* \left\{ \left(\frac{\partial B_{ij}}{\partial \xi}\right)^* + \left(\frac{\partial B_{ij}}{\partial u_k}\right)^* \left(\frac{\partial u_k}{\partial \xi}\right)^* \right\} r_j^* \right], \tag{3.29}$$

$$c_{\eta_p} = \frac{1}{l_i^* A_{ij}^* r_j^*} \left[l_i^* \left\{ \left(\frac{\partial B_{ij}}{\partial \eta_p}\right)^* + \left(\frac{\partial B_{ij}}{\partial u_k}\right)^* \left(\frac{\partial u_k}{\partial \eta_p}\right)^* \right\} r_j^* \right], \tag{3.30}$$

$$c_\omega = \frac{1}{l_i^* A_{ij}^* r_j^*} \left[l_i^* \left(\frac{\partial B_{ij}}{\partial u_k}\right)^* r_k^* r_j^* \right] \tag{3.31}$$

and
$$\phi(t', \eta_\phi) = \frac{1}{l_i^* A_{ij}^* r_j^*} \left[l_i^* \left(\frac{\partial B_{ij}}{\partial u_k}\right)^* g_k(t', \eta_p) r_j^* \right]. \tag{3.32}$$

Equation (3.24) shows that, in the neighbourhood of the critical point $\mathbf{x}^*$, the n dependent variables can be expressed in terms of the arbitrary functions ω and g_i, of which only ω varies significantly with ξ. The value of the functions g_i can be determined by knowledge of the unsteady solution lying outside the small neighbourhood of the plane $\xi = 0$. If we assume the waves to be bounded in space, the functions $\phi(t', \eta_p)$ and $f(t', \eta_p)$ can be taken to be zero. For such disturbances, (3.25) reduces to

$$\frac{\partial \omega}{\partial t'} + (c_\xi \xi_1 + c_\omega \omega) \frac{\partial \omega}{\partial \xi_1} = K\omega, \tag{3.33}$$

where we have used the transformation

$$\xi_1 = \xi + (c_{\eta_p}/c_\xi)\, \eta_p, \eta_p' = \eta_p \tag{3.34}$$

for the spatial co-ordinates. Under this transformation $\partial/\partial \xi_1 = \partial/\partial \xi$ and hence $\partial \omega/\partial \xi_1$ still represents the rate of change of ω in the direction $\mathbf{n}$. Having derived the final equation, it will not be confusing if we drop the prime from t' and the suffix from ξ_1 for the simplicity of the notation. Our approximate equation for the waves in the neighbourhood of the critical point finally takes the form

$$\frac{\partial \omega}{\partial t} + (c_\xi \xi + c_\omega \omega) \frac{\partial \omega}{\partial \xi} = K\omega. \tag{3.35}$$

The co-ordinates η_1 and η_2 can be treated now as parameters and the problem reduces to the discussion of one-dimensional waves governed by (3.35). The expressions for the coefficients c_ξ and K contain $(\partial u_i/\partial x_\alpha)^*$ and hence the approximate equation depends on the basic steady solution (3.2). We may be tempted to eliminate these derivatives by means of a suitable transformation (Bhatnagar

& Prasad 1971) in order to get a single approximate equation valid for the perturbations of an arbitrary steady solution. However, such a transformation is not possible in two- and three-dimensional spaces as K depends not only on $(\partial u_i/\partial\xi)^*$ but also on $(\partial u_i/\partial\eta_p)^*$.

4. Propagation of transonic pulses

From the discussion in § 2, we find that the two assumptions in §3 are satisfied at an arbitrary point $\mathbf{x}^*$ of the sonic surface by the root

$$\lambda = c \equiv n_1 q_1 + n_2 q_2 + n_3 q_3 - a \tag{4.1}$$

provided that we select the normal $\mathbf{n}$ such that

$$n_\alpha = q_\alpha^*/a^*, \quad q_\alpha^* q_\alpha^* = a^{*2}. \tag{4.2}$$

Thus an arbitrary point of the sonic surface is a critical point in the neighbourhood of which a part of the disturbance will stay for a time interval of the order of unity. In this particular case

$$[B_{ij}^*] = \begin{bmatrix} a^* & 0 & 0 & q_1^*/\rho^* a^* & 0 \\ 0 & a^* & 0 & q_2^*/\rho^* a^* & 0 \\ 0 & 0 & a^* & q_3^*/\rho^* a^* & 0 \\ \rho^* a^* q_1^* & \rho^* a^* q_2^* & \rho^* a^* q_3^* & a^* & 0 \\ \rho^* q_1^*/a^* & \rho^* q_2^*/a^* & \rho^* q_3^*/a^* & 0 & a^* \end{bmatrix}, \tag{4.3}$$

$$l_\alpha^* = q_\alpha^*, \quad l_4^* = -1/\rho^*, \quad l_5^* = 0, \tag{4.4}$$

$$r_\alpha^* = q_\alpha^*, \quad r_4^* = -\rho^* a^{*2}, \quad r_5^* = -\rho^*, \tag{4.5}$$

$$c_\omega = \tfrac{1}{2}(\gamma+1)\, a^* \tag{4.6}$$

and

$$c_\xi = \frac{q_i^*}{a^*}\left(\frac{\partial q_i}{\partial \xi}\right)^* - \frac{a^*}{2p^*}\left(\frac{\partial p}{\partial \xi}\right)^* + \frac{a^*}{2\rho^*}\left(\frac{\partial \rho}{\partial \xi}\right)^*. \tag{4.7}$$

Making use of the orthogonal property of the matrix $\mathbf{N}$ and substituting $C \equiv 0$ we get the following expressions for $F_{iv_k}^* r_k^*$ from (3.16) after a few lengthy calculations:

$$F_{\alpha v_j}^* r_j^* = a^*\left(\frac{\partial q_\alpha}{\partial \xi}\right)^* + \frac{1}{\rho^*}\left[\frac{q_\alpha^*}{a^*}\left(\frac{\partial p}{\partial \xi}\right)^* + a_\alpha^{(p)}\left(\frac{\partial p}{\partial \eta_p}\right)^*\right], \tag{4.8}$$

$$F_{4v_j}^* r_j^* = a^*\left(\frac{\partial p}{\partial \xi}\right)^* - \gamma \rho^* a^{*2}\left[\frac{q_\alpha^*}{a^*}\left(\frac{\partial q_\alpha}{\partial \xi}\right)^* + a_\alpha^{(p)}\left(\frac{\partial q_\alpha}{\partial \eta_p}\right)^*\right] \tag{4.9}$$

and

$$F_{5v_j}^* r_j^* = a^*\left(\frac{\partial \rho}{\partial \xi}\right)^* - \rho^*\left[\frac{q_\alpha^*}{a^*}\left(\frac{\partial q_\alpha}{\partial \xi}\right)^* + a_\alpha^{(p)}\left(\frac{\partial q_\alpha}{\partial \eta_p}\right)^*\right] \tag{4.10}$$

From (2.2), (4.4), (4.5) and (4.8)–(4.10) we have

$$l_i^* F_{iv_k}^* r_k^* = a^* q_\alpha^*\left(\frac{\partial q_\alpha}{\partial \xi}\right)^* + \gamma a^{*2}\left[\frac{q_\alpha^*}{a^*}\left(\frac{\partial q_\alpha}{\partial \xi}\right)^* + a_\alpha^{(p)}\left(\frac{\partial q_\alpha}{\partial \eta_p}\right)^*\right] \tag{4.11}$$

and

$$l_i^* A_{ij}^* r_j^* = 2a^{*2}. \tag{4.12}$$

When we transform the derivatives in the second term of (4.11) into derivatives with respect to x_i and use the expression (3.26) for K we get

$$K = -\frac{1}{2a^*} q^*_\alpha \left(\frac{\partial q_\alpha}{\partial \xi}\right)^* - \frac{\gamma}{2}\left(\frac{\partial q_\alpha}{\partial x_\alpha}\right)^*. \tag{4.13}$$

The steady equations corresponding to equations (2.1) give us

$$\left(\frac{\partial q_\alpha}{\partial \xi}\right)^* = -\frac{1}{a^*\rho^*}\left(\frac{\partial p}{\partial x_\alpha}\right)^* \tag{4.14}$$

and

$$\left(\frac{\partial q_\alpha}{\partial x_\alpha}\right)^* = -\frac{1}{\rho^* a^*}\left(\frac{\partial p}{\partial \xi}\right)^* = -\frac{a^*}{\rho^*}\left(\frac{\partial \rho}{\partial \xi}\right)^* = \left(\frac{\partial q}{\partial \xi}\right)^*. \tag{4.15}$$

When we use the relations (4.14) and (4.15) in (4.7) and (4.13), we get

$$K = -c_\xi = -\tfrac{1}{2}(\gamma+1)\,(\partial q/\partial \xi)^*, \tag{4.16}$$

where the quantity $(\partial q/\partial \xi)^*$ represents the space rate of change of the fluid speed at the point $\mathbf{x}^*$ as we move along the streamline in the steady solution and is equal to the acceleration of the fluid element divided by a^*.

Thus the approximate equation governing the propagation of the waves in the neighbourhood of an arbitrary point on the sonic surface is given by a simple equation

$$\frac{\partial \omega}{\partial t} + (c_\omega \omega - K\xi)\frac{\partial \omega}{\partial \xi} = K\omega, \tag{4.17}$$

which depends on the steady flow only through the single parameter K given by (4.16). The perturbations in the flow variables are proportional to the variable ω and are given by

$$\Delta q_\alpha = q^*_\alpha \omega, \quad \Delta p = -\rho^* a^{*2}\omega, \quad \Delta \rho = -\rho^* \omega, \tag{4.18}$$

where

$$\Delta q_\alpha = q_\alpha - q_{\alpha 0}, \quad \Delta p = p - p_0, \quad \Delta \rho = \rho - \rho_0. \tag{4.19}$$

We have derived the equation (4.17) for a general three-dimensional flow which may not be irrotational and in particular we can use it for any two-dimensional or axisymmetric flow. A simple example involving all possible values of the parameter K is a two-dimensional high subsonic flow past an airfoil placed with its axial parallel to the stream. For a suitable contour, it is possible to have a continuous mixed flow with an embedded supersonic region attached to the body as shown in figure 2.

The characteristic equations of the partial differential equation (4.17) are

$$\frac{d\omega}{dt} = K\omega, \quad \frac{d\xi}{dt} = c_\omega \omega - K\xi. \tag{4.20}$$

The general solution of these equations is

$$\omega = \omega_0 e^{Kt}, \quad \xi = \xi_0 e^{-Kt} + (c_\omega/K)\,\omega_0 \sinh Kt, \tag{4.21}$$

where ω_0 and ξ_0 are the values of ω and ξ at $t = 0$ on the characteristic. The sonic point $\omega = 0$, $\xi = 0$ is a singular point of (4.20) and is a saddle point for all possible values of K except $K = 0$. We have shown the phase plane for two different cases:

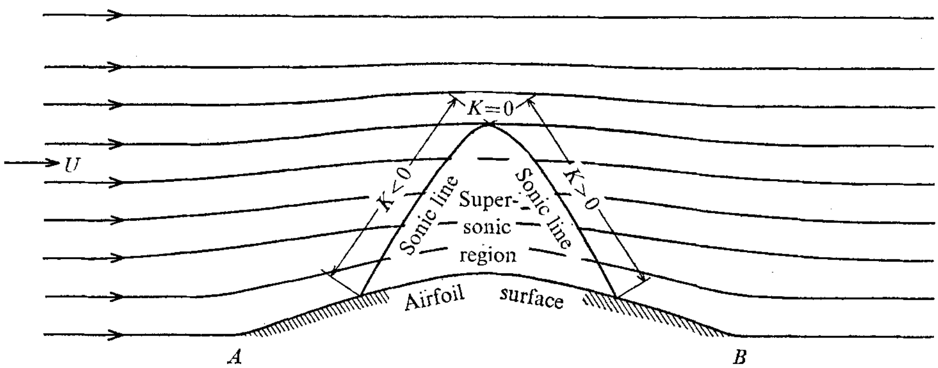

FIGURE 2. The upper half of a high subsonic flow past a symmetrical airfoil.

decelerating flow $K > 0$ and accelerating flow $K < 0$ in figures 3 and 4. The figures have been drawn using non-dimensional variables $\bar{\xi}, \bar{\omega}$, $\bar{K}$ and $\bar{c}_\omega$ defined by

$$\bar{t} = \frac{ta^*}{L}, \quad \bar{\xi} = \frac{\xi}{L\delta}, \quad \bar{\omega} = \frac{\omega}{\delta}, \quad \bar{c}_\omega = \frac{c_\omega}{a^*}, \quad \bar{K} = \frac{KL}{a^*}, \tag{4.22}$$

where L is the length scale of the problem and δ is the non-dimensional first-order small quantity used earlier. A very useful choice of L is $L = a^*/K$ for $K \neq 0$.

The integral curves in the phase plane represent steady solutions of (4.17). The basic undisturbed solution is represented by the line $\omega = 0$. As we are considering perturbations bounded in space, at any instant a perturbation of our basic steady solution can be represented in the phase plane by a closed curve a part of whose boundary is the line $\omega = 0$ as shown by continuous curves in figures 3 and 4. In a perturbation, the space rate of change of ω as we move with the wave velocity $c_\omega \omega - K\xi$ is $K\omega/(c_\omega \omega - K\xi)$, which is also the space rate of change of ω as we move along the integral curves of the characteristic equations. Therefore, during the propagation, the different points of the boundary curve of the disturbance will move along the integral curves of (4.20).

Let S be the area bounded by an arbitrary closed curve in the ξ, ω plane whose points move in accordance with (4.20). As the divergence of the vector field given by the right-hand side of (4.20) is zero, i.e.

$$\frac{\partial}{\partial \xi}(c_\omega \omega - K\xi) + \frac{\partial}{\partial \omega}(K\omega) = 0 = \frac{1}{S}\frac{dS}{dt},$$

it follows that the value of the area S remains constant and equal to its initial value S_0. Thus we get the important pulse area rule (Kantrowitz 1958) that the area occupied by a disturbance in the ξ, ω plane remains constant as the disturbance propagates. The area is conserved even if a weak shock appears in the pulse (Kantrowitz 1958; Landau & Lifshitz 1959, p. 374). Using the rule of constant area, we can follow the complete history of the pulse in the phase plane of (4.20). The following results do not differ in principle from those of Kantrowitz and this shows that the behaviour of sonic pulses in two- or three-dimensional flow does not differ significantly from that of sonic pulses in one-dimensional flow.

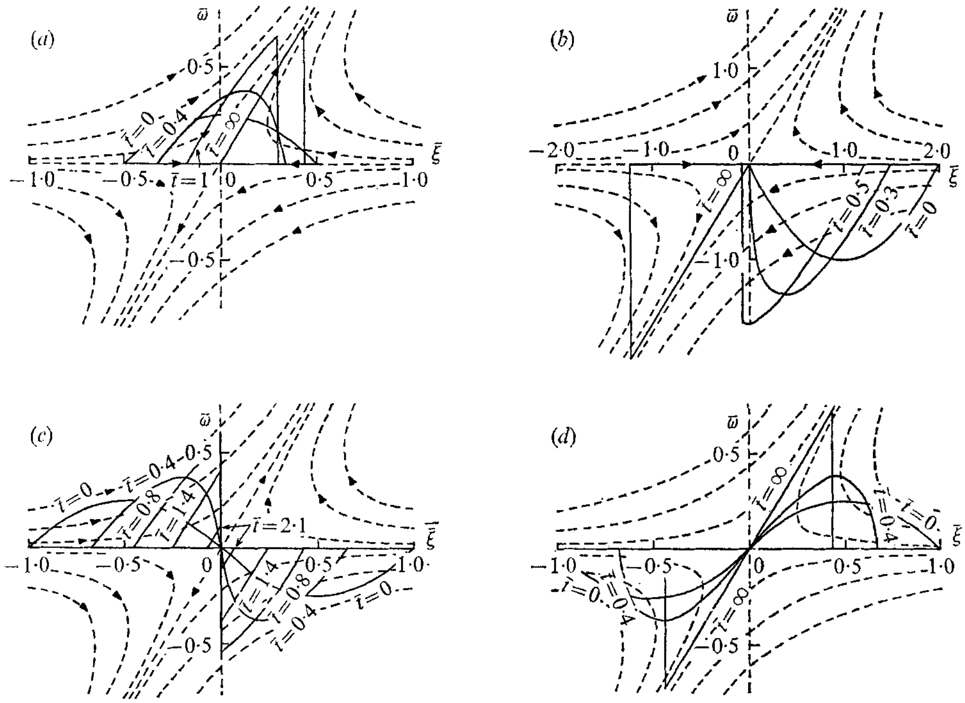

FIGURE 3. Phase plane of the characteristic equations for a decelerating steady flow: $K > 0$; $\bar{K} = 1, \gamma = 1 \cdot 4$. (*a*) A perturbation with positive values of ω attains a triangular form in subsonic region. (*b*) A perturbation with negative values of ω attains a triangular form in the supersonic region. (*c*) Positive and negative areas in two parts of the pulse interact: the pulse weakens and may disappear as in this case. Shock is always at $\xi = 0$. (*d*) Positive and negative areas in two parts of the pulse are trapped separately, changing both upstream and downstream parts of the flow.

We can prescribe the initial shape of the pulse at $t = 0$ by the function

$$\omega = \omega_0(\xi_0). \tag{4.23}$$

At any other time, the shape of the pulse can be obtained in the form

$$\omega = \omega(\xi, t) \tag{4.24}$$

by eliminating the parameter ξ_0 from (4.21) and (4.23). As the pulse propagates, the slope $\partial\omega/\partial\xi$ at any point moving with the pulse changes and we can easily calculate the slope at any time in terms of the initial slope $d\omega_0/d\xi_0$ from equations (4.21):

$$\frac{\partial\omega}{\partial\xi} = \frac{d\omega_0}{d\xi_0}\bigg/\left\{e^{-2Kt}\left(1 - \frac{c_\omega}{2K}\frac{d\omega_0}{d\xi_0}\right) + \frac{c_\omega}{2K}\frac{d\omega_0}{d\xi_0}\right\}. \tag{4.25}$$

Now we discuss three particular cases separately.

Case 1. *Flows decelerating through the speed of sound, $K > 0$*

Equation (4.25) shows that the slope $\partial\omega/\partial\xi$ at a point moving with the pulse becomes infinite at some time if initial slope $d\omega_0/d\xi_0 < 0$, or tends to a limiting value

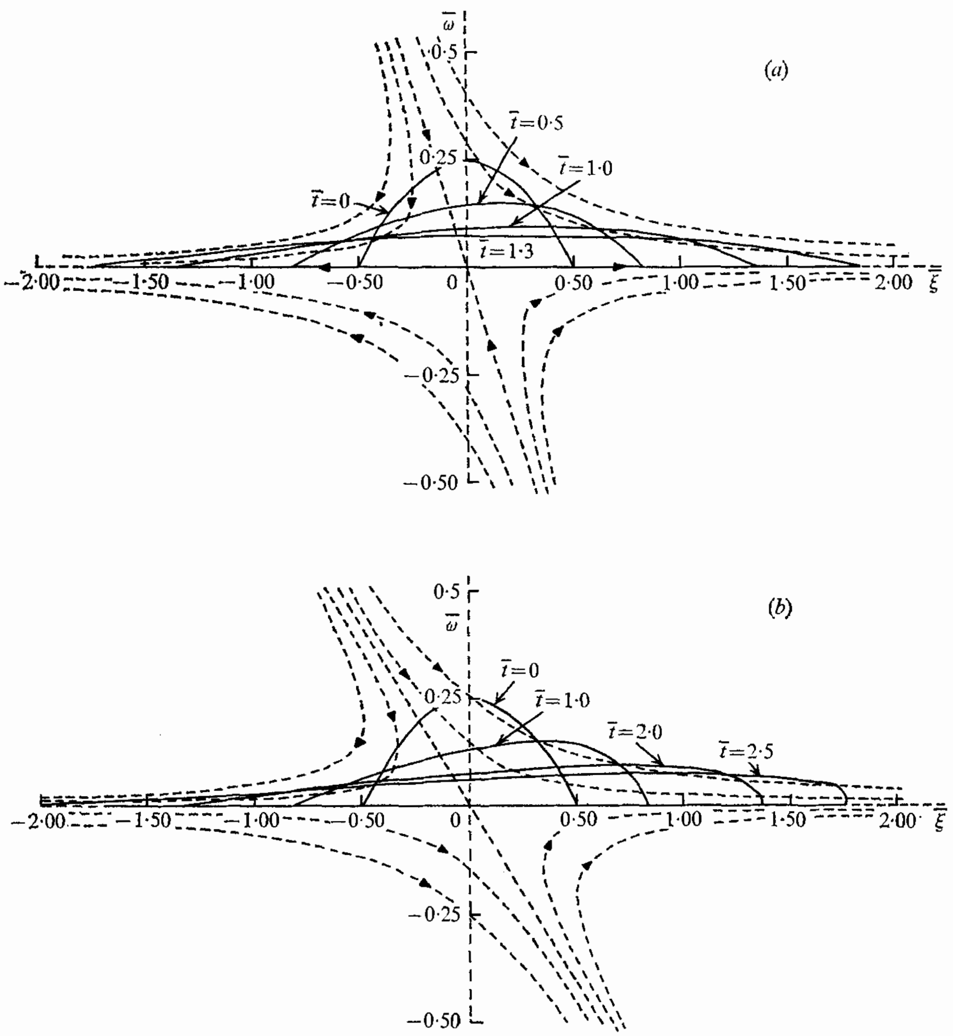

FIGURE 4. Phase plane of the characteristic equations for an accelerating steady flow; $K < 0$, $\gamma = 1{\cdot}4$. (*a*) $K = -1$. Acceleration at the critical point is sufficiently large and therefore a continuous pulse remains continuous as it vanishes from the transonic region. (*b*) $K = -0{\cdot}5$. Acceleration at the critical point is small and a shock appears in the compression region before the disturbance vanishes from the transonic region.

$2K/c_\omega$ if $d\omega_0/d\xi_0 > 0$. Therefore, a shock wave always appears in a continuous pulse at a time

$$T = \frac{1}{2K}\ln\left[\left\{\frac{c_\omega}{2K}\left(\frac{d\omega_0}{d\xi_0}\right)_{\min} - 1\right\}\Big/\left\{\frac{c_\omega}{2K}\left(\frac{d\omega_0}{d\xi_0}\right)_{\min}\right\}\right], \tag{4.26}$$

where $(d\omega_0/d\xi_0)_{\min}$ is the smallest value of the negative slope in the pulse. The shock appears in the interior if the pulse has a point of inflexion in the compression region. The motion of the shock can be easily followed by using the result that for a weak shock its velocity is the mean of the characteristic velocities

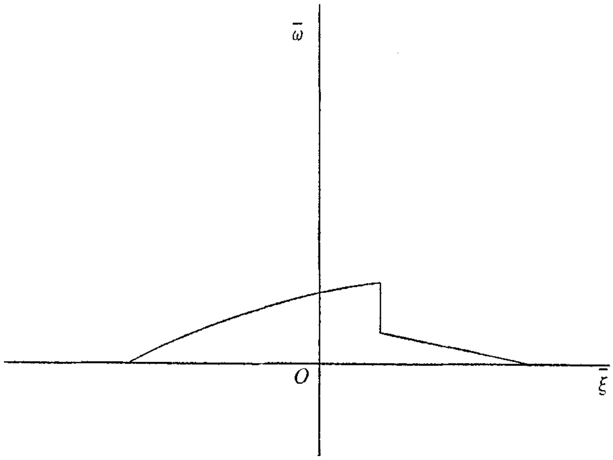

FIGURE 5. Shape of the pulse with positive ω for some intermediate time when there is initially a point of inflexion in the compression region.

$c_\omega \omega_a - K\xi_s$ just ahead and $c_\omega \omega_b - K\xi_s$ just behind the shock. Thus the shock position ξ_s satisfies

$$d\xi_s/dt = -K\xi_s + \tfrac{1}{2}c_\omega\{\omega_a(\xi_s, t) + \omega_b(\xi_s, t)\}. \tag{4.27}$$

Equations (4.25) and (4.21) show that the slope reaches its limiting value as $\exp(-2Kt)$ tends to zero but a general point of the phase plane tends to infinity (or the origin) as $\exp(Kt)$ (or $\exp(-Kt)$). Therefore, the slope reaches its limiting value faster than a characteristic approaches the critical point or infinity. Therefore, after sufficiently long time, any disturbance will assume a triangular shape bounded at its leading edge by a shock wave. When the initial shape contains a point of inflexion in the region where the slope is negative, the shape of the pulse for some intermediate time will be as shown in figure 5 with a shock in the interior of the pulse. Since the area of the pulse remains constant, a close examination of the phase plane shows that the pulse will ultimately become stationary with two sides bounded by the straight lines $\omega = 0$ and $\omega = (2K/c_\omega)\,\xi$ and the third side bounded by a straight line parallel to $\xi = 0$ representing a shock wave. The distance of the shock from the origin can be easily obtained by equating the area of the triangle with the initial area. In figure 3 we have shown the history of four types of disturbances. If ω is positive the pulse is trapped in the subsonic part of the flow, changing it partially into a supersonic flow. On the other hand, if ω is negative it is trapped in the supersonic region, changing it partially into a subsonic one. A numerical evaluation of the pulse shape using (4.21) and a numerical integration of (4.27) from the formation of the shock for an initially parabolic pulse shows that ξ_s approaches its limiting value a long time after the pulse has attained a triangular shape.

Case 2. Flows accelerating through the speed of sound, $K < 0$

Equation (4.25) shows that the slope $\partial\omega/\partial\xi$ tends to zero as t tends to infinity where $d\omega_0/d\xi_0 > -2|K|/c_\omega$ and tends to infinity where $d\omega_0/d\xi_0 < -2|K|/c_\omega$. Therefore, a continuous pulse will remain continuous if the initial shape is not very steep in the compression region ($d\omega_0/d\xi_0 < 0$). A shock appears at time T given by (4.26) when the shape is sufficiently steep in the compression region so that $d\omega_0/d\xi_0 < -2|K|/c_\omega$ is satisfied.

Examination of the phase plane shows that the amplitude of the pulse continuously decreases and that both ends of the pulse move away from the sonic point. Of course, the area occupied by the pulse remains constant. The disturbance decays and ultimately vanishes from the transonic region. The history of such disturbances has been shown in figure 4.

Case 3. $K = 0$

In the case of high subsonic flows past bodies, an embedded supersonic flow appears and there exists one curve on the sonic surface (a point on the sonic curve in two-dimensional flows, see figure 2) where the streamlines are tangential to the sonic surface. If x^* is a point on such a curve, the fluid particles attain a maximum velocity here and $K = 0$. Equation (4.17) reduces to

$$\frac{\partial \omega}{\partial t} + c_\omega \omega \frac{\partial \omega}{\partial \xi} = 0. \tag{4.28}$$

In the phase plane the characteristic equations reduce to a family of straight lines parallel to $\omega = 0$. If we consider an initially continuous pulse, the amplitude remains constant until the shock wave which appears at a time

$$T = -\{c_\omega (d\omega_0/d\xi_0)_{\min}\}^{-1}$$

in the compression region has been overtaken by the maximum (or minimum) value of ω. The trailing front remains fixed at its initial position and the leading front bounded by a shock wave moves farther and farther away from the trailing front. The disturbance ultimately decays and vanishes from the transonic region as shown in figure 6.

Thus we conclude that a transonic flow is stable with respect to small disturbances everywhere except in the neighbourhood of the points where the flow is strictly decelerating. In the second case the flow is neutrally stable in the sense that the disturbance is trapped and attains a stationary position. In real flows disturbances will be continuously fed into the transonic region but, since the disturbances will be random in nature, we expect the positive and negative areas of the trapped waves to be almost equal, leading to the presence of weak shocks in decelerating portions of the transonic flow. Our analysis does not really contradict the conclusions of Nieuwland & Spee that a continuously decelerating transonic flow can be regarded as stable for all practical purposes, however, we cannot get a continuously decelerating flow which is completely free from weak shocks. Whenever the boundary conditions at the aerofoil or in the free stream are changed slightly from the theoretical value, disturbances of the same sign in area will be continuously created and the flow will be partly replaced by an accelerating flow which is either terminated by a shock or headed by a shock with strength depending on the deviation of the boundary condition. This is clearly shown from the results of experiments mentioned by Spee. Spee has shown that two-dimensional turning effect is important, at least for waves originating from

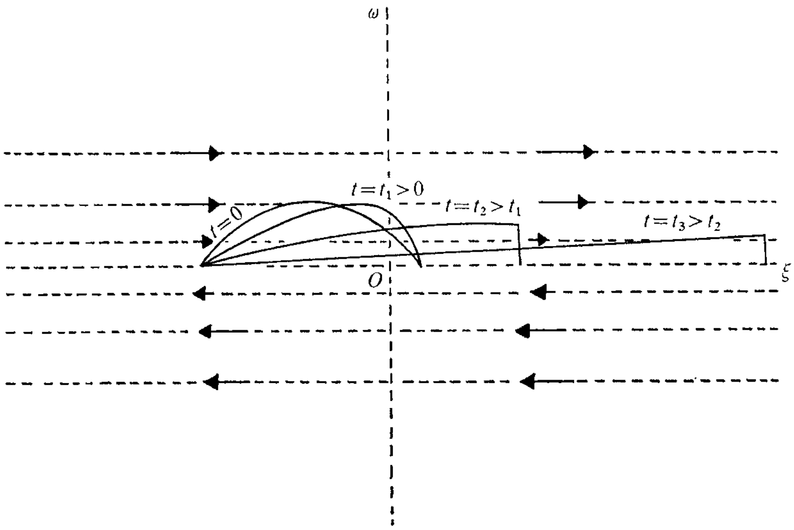

FIGURE 6. $K = 0$. A disturbance with positive values of ω first deforms with constant amplitude until a shock is formed which starts moving to the right. The pulse ultimately vanishes from the sonic region.

regions away from the sonic line and, therefore, in order to get a more realistic local picture of the transonic waves we must improve the present theory so that it takes account of the real two- and three-dimensional effects, and this does not seem to be difficult. We shall be able to present an improved form of the theory in near future. The streamlines normal to the direction of the wave front play a very important role. For flows past convex bodies they are generally pushed together where the flow is accelerating and in this region they will have a tendency to converge (figure 2). This will lead to some increase in the amplitude of the pulse. In the region where the flow is decelerating, they have a tendency to diverge from each other and this will result in a decrease in the amplitude of the pulse.

In the work of Kuo on two-dimensional transonic flows, the curvature of the body appears in the result but he concludes that the curvature of the body plays no decisive role in the question of stability. This is also evident from our analysis as it is completely independent of the curvature of the body. The effect of curvature should appear through the convergence or divergence of the streamlines.

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